

THE SEN CONJECTURE FOR FUNDAMENTAL MONOPOLES OF DISTINCT TYPES

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ABSTRACT

I exhibit a middle-dimensional square integrable harmonic form on the moduli space of distinct fundamental BPS monopoles of an arbitrary Lie group. This is in accord with Sen's S-duality conjecture. I also show that the moduli space has no closed or bound geodesics

I. Introduction

There have recently been some advances in our understanding of the interactions of BPS monopoles in the case that a compact semi-simple Lie group G of rank k breaks to its maximal torus $C(G) \cong U(1)^k$ by a Higgs field Φ in the adjoint representation [1,2,3,4]. In the simplest non-trivial case ($G = SU(3)$) the metric on the relative moduli space has been identified as the Taub-NUT metric and a square integrable self-dual harmonic form exhibited which is consistent with Sen's S-Duality conjecture [5]. A proposal has been made [4] for the metric on the general moduli space of distinct fundamental monopoles but the corresponding middle-dimensional harmonic form was not found. In this paper I shall remedy that deficiency by giving a simple explicit expression for this form which represents a bound state of a system of fermions and monopoles in the $n = 4$ supersymmetric version of the theory. I shall also show that in distinction to the case of arbitrary numbers of identical $SU(2)$ monopoles, there are no classical closed geodesics. Indeed there are

no classical bound orbits at all.

Fundamental Monopoles

Monopoles in this theory may carry k types of magnetic charge. One may always arrange, by means of a conjugation if necessary, that the vacuum expectation value of the Higgs field at infinity Φ_∞ lies in the Cartan sub-algebra \mathfrak{h} . Associated with Φ_∞ is a hyperplane in \mathfrak{h} and a unique set of k simple positive roots β_a , $a = 1, 2, \dots, k$. A general monopole has an associated magnetic charge vector \mathbf{g} taking values in \mathfrak{h}^* , the dual of the Cartan sub-algebra. In fact the Dirac quantization condition dictates that \mathbf{g} lies on a lattice spanned by the reciprocal vectors

$$\beta_a^* = \frac{\beta_a}{\beta_a \cdot \beta_a}, \quad (1)$$

of the positive simple roots, that is

$$\mathbf{g} = \frac{4\pi}{e} \sum_{a=1}^{a=k} n_a \beta_a^*, \quad (2)$$

where e is the gauge coupling constant and $\{n_a\}$ are integers.

There are k types of *fundamental* monopoles corresponding to an embedding of an $SU(2)$ monopole and each has unit magnetic charge with respect to one of the k circle subgroups of $C(G)$ and zero magnetic charge with respect to the other $k - 1$ circle subgroups. The a 'th fundamental monopole is associated with the dual root β_a^* and has positive mass

$$m_a = \frac{4\pi}{e} \beta_a^* \cdot \Phi_\infty. \quad (3)$$

Remarkably a sort of superposition of distinct fundamental monopoles is possible. These correspond to a composite dual root

$$\alpha^* = \sum_{a=1}^{a=k} n_a \beta_a^* \quad (4)$$

and have mass

$$m = \sum_{a=1}^{a=k} n_a m_a, \quad (5)$$

where the integers n_a are non-negative.

We shall consider the moduli space of n distinct fundamental monopoles corresponding to n positive roots α_i . The n positive roots define a Dynkin diagram. The asymptotic forces between the monopoles are given entirely in terms of the inner products $\alpha_i \cdot \alpha_j \neq 0$. Only roots which are connected in the Dynkin diagram (i.e. for which $\alpha_i \cdot \alpha_j \neq 0$) interact and thus one is led to restrict attention to Dynkin diagrams which are connected and therefore the α_i constitute the roots of (possibly smaller) group. The Dynkin diagram has just $n - 1$ links corresponding to the number of unordered pairs (i, j) for which $\alpha_i \cdot \alpha_j \neq 0$. We shall use the index A to label these links. Physically the links correspond to $n - 1$ relative position vectors $\mathbf{r}_A = \mathbf{x}_i - \mathbf{x}_j$ and $n - 1$ relative phases $\psi_A \in (0, 4\pi]$. For each link we define the positive numbers

$$\lambda_A = -2\alpha_i^* \cdot \alpha_j^*. \quad (6)$$

The moduli space.

In general the moduli space \mathcal{M}_n of n BPS monopoles is known to be a $4n$ -dimensional geodesically complete HyperKähler manifold. Because the centre of mass motion may be factored out, it is of the form

$$\mathcal{M}_n \cong \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}_{n-1}^{\text{rel}}}{\mathcal{D}} \quad (7)$$

where the relative moduli space $\mathcal{M}_{n-1}^{\text{rel}}$ is a geodesically complete $4(n-1)$ -dimensional hyperKähler manifold and the group \mathcal{D} is a discrete normal subgroup of the isometry group of $\mathcal{M}_{n-1}^{\text{rel}}$. The isometry group contains a copy of $SO(3)$ acting on the the three complex structures as a triplet. In general $\mathcal{M}_{n-1}^{\text{rel}}$ will have no additional exact continuous isometries but for large separation one may identify as coordinates $n - 1$ relative cartesian positions \mathbf{r}_A and $n - 1$ and certain angles θ_A as coordinates on $\mathcal{M}_{n-1}^{\text{rel}}$. One then finds that asymptotically there is an additional approximate triholomorphic action of the torus group T^{n-1} corresponding to shifting the angles. The associated Killing vector fields are

$$K^{A\alpha} \frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial \theta_A} \quad (8)$$

with $\alpha = 1, 2 \dots 4n - 4$. The invariance corresponds physically to the conservation of the $n - 1$ relative electric charges which may be carried by dyons. One may

explicitly write down the asymptotic metric $g_{\alpha\beta}$ in terms of magnetic charges of the monopoles [6,7]. In the rest of this section I will describe some general properties of the asymptotic metric. In the case of distinct monopoles the asymptotic metric is believed to be exact and so these properties are shared by the exact metric. Thus for example there is always an asymptotic identification between the space of relative positions and the quotient of $\mathcal{M}^{\text{red}} \equiv M_{n-1}^{\text{rel}}/T^{n-1}$. This identification is believed to be exact in the case of distinct monopoles. The cartesian coordinates \mathbf{r}_A are, up to a scale, the three moment maps corresponding to the triholomorphic Killing fields $\frac{\partial}{\partial\theta_A}$. The torus action on $\mathcal{M}_{n-1}^{\text{rel}}$ is not even locally hypersurface orthogonal and so regarding $\mathcal{M}_{n-1}^{\text{rel}}$ as a T^{n-1} bundle over \mathcal{M}^{red} one gets a non-trivial torus connection in the standard Kaluza-Klein fashion by taking the horizontal subspaces to be orthogonal to the torus fibres. Thus a curve $x^\alpha(t)$ in M_{n-1}^{rel} is horizontal if

$$K^{A\alpha}g_{\alpha\beta}\frac{dx^\beta}{dt}=0. \quad (9)$$

The curvature 2-forms F_A on M_{n-1}^{rel} are given in terms of the Killing co-vector fields $K_\alpha^A = g_{\alpha\beta}K^{A\beta}$ by

$$F^A = \partial_\alpha K_\beta^A - \partial_\beta K_\alpha^A. \quad (10)$$

It is a well known elementary deduction from Killing's equations that if a metric $g_{\alpha\beta}$ is Ricci flat then a two-form obtained by taking the exterior derivative of the one-form obtained by lowering the index of any Killing vector field is both closed and co-closed, that is it satisfies Maxwell's equations. Since a HyperKähler metric is necessarily Ricci flat it follows that we obtain $n-1$ solutions of Maxwell's equations in this way. They will be important later.

Any $SO(3) \times T^{n-1}$ invariant metric, whether HyperKähler or not, may locally be cast in the form

$$ds^2 = G_{AB}d\mathbf{r}_A \cdot d\mathbf{r}_B + H^{AB}(d\theta_A + \mathbf{W}_{AC} \cdot d\mathbf{r}_C)(d\theta_B + \mathbf{W}_{BD} \cdot d\mathbf{r}_D), \quad (11)$$

where G_{AB} , H^{AB} and \mathbf{W}_{AC} are independent of the angles θ_A . The Maxwell vector potentials are then given by

$$K^A = H^{AB}(d\theta_B + \mathbf{W}_{BD} \cdot d\mathbf{r}_D). \quad (12)$$

In the particular case that the metric is HyperKähler one has (among other things) that

$$G_{AB} = H_{AB} \quad (13)$$

where $H_{AB}H^{BC} = \delta_A^C$.

Taub-NUT Space

It may be helpful to begin by considering the simplest example when $G = SU(3)$ and $n = 2$. In this case M_1^{rel} has been identified [1,2,3] as the complete self-dual Taub-NUT space on $\mathbb{R}^4 \equiv \mathbb{H}$, with the isometry group $U(2)$ acting on \mathbb{H} by left multiplication by a unit quaternion and right multiplication by a circle subgroup of the quaternions. Each of the two-sphere's worth of complex structures act by left multiplication by a unit quaternion and the circle action commutes with this action, i.e. it is triholomorphic. The $SO(3)$ action however rotates the complex structure. If $\sigma_1, \sigma_2, \sigma_3$ are left-invariant one-forms on $SU(2)$ the metric takes the form

$$ds^2 = V^{-1}4M^2\sigma_3^2 + V\left(dr^2 + r^2(\sigma_1^2 + \sigma_2^2)\right) \quad (14)$$

$$= V^{-1}(d\tau + \boldsymbol{\omega} \cdot d\mathbf{r})^2 + V d\mathbf{r} \cdot d\mathbf{r} \quad (15)$$

with $V = 1 + \frac{2M}{r}$, $\nabla \times \boldsymbol{\omega} = \nabla V$ and $d\tau + \boldsymbol{\omega} \cdot d\mathbf{r} = 2M\sigma_3 = 2M(d\psi + \cos\theta d\phi)$, where (ψ, θ, ϕ) are Euler angles and M is a positive constant. The metric has a coordinate singularity at $r = 0$ because the isometry group has a fixed point at the origin, sometimes referred to as a *NUT*. As stated above the three cartesian coordinate functions \mathbf{r} which parameterize the orbits of the triholomorphic circle action may be invariantly characterized (possibly up to an overall scale) as its three moment maps.

Because it is the exact metric the manifold must be complete and therefore the mass parameter M which appears in the Taub-NUT metric must, as stated above be *positive* in distinction to the asymptotic form of the relative moduli space of two identical $SU(2)$ monopoles for which the mass parameter is negative. This change of sign reflects the different nature of the interactions between monopoles of the same type and those of different types. In general the former are attractive and the latter are repulsive. As we shall see later this leads to significant differences in the behaviour of geodesics and bound states.

It is worth remarking here that although the negative mass Taub-NUT metric is not complete it is not really pathological and has as good a geometrical pedigree as the positive mass metric. The problem is that the signature of the metric changes from $++++$ to $----$ as one passes $r = 2|M|$. The region inside $r = 2|M|$ is complete near $r = 0$, which is just a coordinate singularity. To understand the change of signature at $r = 2|M|$ recall that the Taub-NUT metric with positive mass may be obtained as a hyperKähler quotient of the flat HyperKähler metric on \mathbb{R}^8 . The zero set of the relevant moment map is a smooth 5-dimensional submanifold in \mathbb{R}^8 . One then quotients by the \mathbb{R} -action generated by the moment maps. To get the Taub-NUT metric with negative mass one starts with the flat pseudo-HyperKähler metric on $\mathbb{R}^{4,4}$. The zero set of the relevant moment map is again a smooth 5-dimensional submanifold but now in $\mathbb{R}^{4,4}$. The metric induced on it from the pseudo-euclidean metric of signature $++++--$ has signature $++++-$ if $r > 2|M|$ and signature $----+$ if $r < 2|M|$. The orbits of the \mathbb{R} -action generated by the moment maps are timelike if $r > 2|M|$ and spacelike if $r < 2|M|$. As a consequence the quotient has signature $++++$ if $r > 2|M|$ and $----$ if $r < 2|M|$. Of course the induced metric vanishes on $r = 2|M|$.

To return to the positive mass case: an $SO(3)$ -invariant square integrable 2-form F exists in Taub-NUT which is both closed and co-closed [2,3]. In fact the two-form found in [2,3] is *precisely* that obtained from the $U(1)$ Killing field. It is exact,

$$F = dA, \quad (16)$$

and the globally well-defined one-form A_α is related to the Killing vector field $K^\alpha \frac{\partial}{\partial x^\alpha}$ which generates the tri-holomorphic circle action by index lowering with the metric $g_{\alpha\beta}$

$$A_\alpha = g_{\alpha\beta} K^\beta. \quad (17)$$

Explicitly

$$A = 4M^2 V^{-1} \sigma_3. \quad (18)$$

Because $V \rightarrow 1$ at large r , the length of the orbits of the circle action, which is proportional to the norm of A_α , tends to constant at infinity. Therefore the one form A is not square integrable.

The General Moduli-space Metric

Lee, Weinberg and Yi [4] have proposed in the case of $n \leq k$ fundamental

monopoles for any group G that the asymptotic moduli space metric at large separations is in fact exact for all separations. As explained above there is no loss of generality in taking $n = k$. The metric is specified by

$$G_{AB} = \mu_{AB} + \frac{g^2 \lambda_A \delta_{AB}}{8\pi r_A} \quad (19)$$

where $r_A = |\mathbf{r}_A|$ and $g = \frac{4\pi}{e}$ and μ_{AB} is a constant positive definite non-diagonal symmetric matrix. The angles θ_A are given in terms of the phases $\psi_A \in (0, 4\pi]$ by

$$\theta_A = \frac{g^2 \lambda_A}{8\pi} \psi_A \quad (20)$$

where the λ_A are given by (6) and

$$\mathbf{W}_{AC} = \frac{g^2 \lambda_A \delta_{AC}}{8\pi} \mathbf{w}(\mathbf{r}_A), \quad (21)$$

where $\mathbf{w}(\mathbf{r})$ is the vector potential of a single Dirac monopole. Note that the repeated indices in (21) are not summed over and note also that there is a single radial magnetic field \mathbf{B}_A associated with each link.

Lee, Weinberg and Yi have checked that the apparent singularities arising when one or more relative distances r_A vanishes is a coordinate artefact and that the metric is in fact complete there. This is certainly a necessary test of their conjecture. They did not discuss the global topology. In the case $G = SU(k+1)$ it is known from other arguments [1] that topologically $\mathcal{M}_{k-1}^{\text{rel}} \cong \mathbb{R}^{4k-4}$ and it seems likely that this is true in general. This agrees with a count of the fixed points of the $U(1)$ actions. One might worry that the proposed metric is just the metric product of $k-1$ copies of the Taub-NUT metric but this appears not to be the case because the matrix μ_{AB} and hence the matrix G_{AB} is not diagonal.

Harmonic Forms

The square integrable harmonic two form on the Taub-NUT metric has an obvious analogue on the Lee-Weinberg-Yi metric. As pointed out above each of the $k-1$ $U(1)$ Killing vector fields $K^{A\alpha}$ provides a smooth, exact and co-closed $SO(3)$ - invariant two form F^A . This dies away like $\frac{1}{r_A^2}$ at infinity. The volume is easily seen to grow as

$$\prod_{A=1}^{A=k-1} r_A^3 \quad (22)$$

at infinity and therefore F^A is not square integrable. We can obtain higher rank closed and co-closed even-dimensional forms by taking sums of exterior products but it is clear that there is only one way of constructing a square integrable form in this way : one must take the middle-dimensional form:

$$F = \prod_{A=1}^{A=k-1} F^1 \wedge F^2 \dots \quad (23)$$

It is easy to see that although the $2(k-1)$ -form F is exact, the $(2k-3)$ -form of which it is the exterior derivative is not square integrable. Finally it is clear that this middle dimensional form, which is manifestly $SO(3) \times T^{k-1}$ invariant, is self-dual, in other words

$$F = \star F \quad (24)$$

where \star is the Hodge star operation on forms. Thus the $2(k-1)$ -form F satisfies all the properties predicted by S-duality whose construction was left as an outstanding problem by Lee, Weinberg and Yi.

To be strictly accurate there is an issue of uniqueness. In fact there is, as far as I know, no rigorous proof that the square integrable harmonic form on the relative moduli space of two identical $SU(2)$ monopoles is unique. The arguments given in [5] assume $SO(3)$ invariance and while it is clear by averaging that if a harmonic form exists there also exists an $SO(3)$ invariant one it is not obvious without a further argument that every harmonic form is $SO(3)$ invariant. A similar statement is true in the present case with respect to $SO(3) \times T^{k-1}$ invariance.

To check uniqueness among non- $SO(3) \times T^{k-1}$ -invariant forms one might proceed by considering the difference between F and a putative rival G say. One has

$$F - G = dB \quad (25)$$

for some globally defined $2k-3$ -form B which satisfies

$$\delta dB = 0. \quad (26)$$

Contracting with B and integrating over the part of $\mathcal{M}_{k-1}^{\text{rel}}$ inside a large bounding hypersurface $\partial\mathcal{M}_{k-1}^{\text{rel}}$ gives

$$\int_{\mathcal{M}_{k-1}^{\text{rel}}} ||dB||^2 = \int_{\partial\mathcal{M}_{k-1}^{\text{rel}}} B\delta B. \quad (27)$$

The area of the boundary $\partial\mathcal{M}_{k-1}^{\text{rel}}$ increases as

$$\prod_{A=1}^{A=k-1} r_A^2. \quad (28)$$

Evidently if B and δB decrease faster than

$$\prod_{A=1}^{A=k-1} r_A^{-1}, \quad (29)$$

then we can deduce uniqueness: $dB = F - G = 0$. It is not difficult to convince one's self that these fall-off conditions are very plausible but I have no truly rigorous proof.

Absence of Bound Orbits and the Virial Theorem

The original reason for being interested in the metric on the moduli space is that the geodesics give an approximate description of the slow motion of monopoles. At large separations this is in fact already known from a consideration of the forces between widely separated monopoles. Indeed that is how the asymptotic metric is constructed! In the case of fundamental monopoles the asymptotic metric is exact and thus use of the metric could be avoided if one wishes. The motion of the monopoles remains of interest however. From a physical point of view it is simplest to consider the projections of geodesics onto the quotient space \mathcal{M}^{red} and to ignore the motion in the torus fibres. The effect of the latter is to endow the magnetic monopoles with conserved electric charges

$$Q^A = K_\alpha^A \frac{dx^\alpha}{dt} = H^{AB}(\dot{\theta}_B + \mathbf{W}_{BC} \cdot \mathbf{v}_C), \quad (30)$$

where $\mathbf{v}_C = \frac{d\mathbf{x}_C}{dt}$ is the velocity in the reduced space \mathcal{M}^{red} . Given an orbit in the reduced space we can reconstruct the motion in the angles θ_A .

In the case of Taub-NUT the reduced motion has a very simple description [8]. Angular momentum conservation implies that the orbits lie on a cone centred on the origin. The existence of a generalization of the conserved Lagrange-Laplace-Runge-Lenz vector for the Coulomb problem then implies that the orbits lie in a plane. As a consequence they are conic sections. If the mass parameter is positive, as it is

for two distinct fundamental $SU(3)$ monopoles, there are only hyperbolic orbits. If the mass parameter is negative as it is for the approximate metric for two identical $SU(2)$ monopoles then bound elliptical orbits are possible. The corresponding bound geodesics persist on the exact metric which is also known to support closed geodesics [9]. Recently Bielawski [10] has given a totally geodesic embedding of the strongly centred $SU(2)$ monopole moduli space of charge 2 into the strongly centred $SU(2)$ moduli space of charge n thereby showing the existence of closed geodesics for all n .

The reason for this difference is essentially because $SU(2)$ dyons are oppositely charged with respect to the same $U(1)$ which gives rise to attractive forces between the dyons. In the case of distinct fundamental $SU(3)$ monopoles the dyons are charged with respect to different $U(1)$'s and in fact the forces are repulsive.

It is clearly of interest to ask whether there are bound orbits, and hence bound geodesics, for more than two distinct fundamental monopoles. In fact there are not. To prove this we establish a simple generalization of the Virial Theorem. We can obtain our result directly from the geodesic equations of motion but some extra insight is afforded by constructing an effective Lagrangian L from which to obtain the equations of motion of the orbits with fixed electric charges on \mathcal{M}^{red} . The Lagrangian (which is *not* just obtained by substituting the expressions for conserved charges into the action for geodesics) is readily seen to be given by

$$L = \frac{1}{2}G_{AB}\mathbf{v}_A \cdot \mathbf{v}_B - \frac{1}{2}H_{AB}Q^A Q^B + Q^A \mathbf{W}_{AB} \cdot \mathbf{v}_B. \quad (31)$$

The conserved energy is

$$E = \frac{1}{2}G_{AB}\mathbf{v}_A \cdot \mathbf{v}_B + \frac{1}{2}H_{AB}Q^A Q^B. \quad (32)$$

The two terms in (32) correspond to the kinetic energy

$$T = \frac{1}{2}G_{AB}\mathbf{v}_A \cdot \mathbf{v}_B, \quad (33)$$

which is positive definite and the potential energy

$$V = \frac{1}{2}H_{AB}Q^A Q^B, \quad (34)$$

which is also positive and repulsive for distinct fundamental monopoles. It follows immediately that the relative distance r_A can only vanish if the associated electric charge Q^A vanishes. Because magnetic fields do no work, the vector potentials $\mathbf{W}_B = Q^A \mathbf{W}_{AB}$ are absent from the expression for the energy. The equations of motion are

$$-\frac{d}{dt}(G_{AB}\mathbf{v}_B) + \frac{\partial T}{\partial \mathbf{r}_A} - \frac{\partial V}{\partial \mathbf{r}_A} - \mathbf{B}_A \times \mathbf{v}_A = 0. \quad (35)$$

Note that although the index B is summed over, there is no sum over the index A . One now takes the dot product with \mathbf{r}_A . Because of the special form of the vector potential we are considering the last term vanishes. Thus after a slight re-arrangement (again with a sum over B but not A)

$$\frac{d}{dt}(G_{AB}\mathbf{v}_B \cdot \mathbf{r}_A) = G_{AB}\mathbf{v}_B \cdot \mathbf{v}_A + \mathbf{r}_A \cdot \frac{\partial T}{\partial \mathbf{r}_A} - \mathbf{r}_A \cdot \frac{\partial V}{\partial \mathbf{r}_A}. \quad (36)$$

At this stage (although one loses some information) it is quickest to sum over A and use the homogeneity properties with respect to the \mathbf{r}_A of the kinetic and potential energies. One has

$$T = T_0 + T_{-1} \quad (37)$$

and

$$V = V_0 + V_{-1} \quad (38)$$

where $T_0 = \frac{1}{2}\mu_{AB}\mathbf{v}_A \cdot \mathbf{v}_B$ and $V_0 = \frac{1}{2}\mu_{AB}Q^A Q^B$ are positive and independent of position (V_0 is just a constant) and T_{-1} is homogeneous of degree -1 in the \mathbf{r}_A 's and V_{-1} is also positive and homogeneous of degree -1 in the \mathbf{r}_A 's. Thus

$$\frac{d}{dt}(G_{AB}\mathbf{v}_B \cdot \mathbf{r}_A) = 2T_0 + T_1 + V_{-1} = E' + T_0, \quad (39)$$

where

$$E' = E - V_0 = T + V_{-1} \quad (40)$$

is a strictly positive constant. One now integrates (39) from an initial time t_i to a final time $t_f = t_i + P$ and divides by P . If one has a bound orbit then the right hand side must go to zero for large enough P but the right hand side is never smaller than the positive constant E' . This is a contradiction. This argument also rules out periodic orbits but in that case, of course, one could choose P to be the period of

the orbit. If the forces were attractive and if bound orbits existed we would obtain in this way a relation between the average kinetic and potential energies. This is a generalization of the usual Virial Theorem.

The non-existence of bound classical motions indicates that there are no purely bosonic quantum bound states, in distinction to the $SU(2)$ case where it is known that there are non-BPS bound states [7]. Thus the existence of the Sen bound state in the case of distinct fundamental monopoles is a more unexpected phenomenon and appears to owe its existence to a more subtle consequence of the fermionic structure of the theory.

Acknowledgement

I would like to thank Nick Manton for helpful conversations and a critical reading of the manuscript.

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